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VARIATIONAL METHOD OF CRACK-CONTOUR LOCATION FOR THREE-DIMENSIONAL PROBLEM WITH UNILATERAL CONSTRAINTS

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Extremal properties are established for the solution of the problem of cohesionless normal-rupture crack formation: namely, that the true contour of a Christianovich crack corresponds to the maximum volume of the cavity. Examples of the application of this principle are considered.

Mathematical Model of Christianovich Crack in an Elastic Body

In an elastic space compressed at infinity by a uniform stress σ, acting perpendicularly to the plane $S : z = 0$, forces symmetric with respect to S but in the opposite direction are applied, leading to the formation of normal rupture over a certain part of this plane of maximum tensile stress (breakdown). If the effect of the cohesive forces of the material over the S plane may be neglected in comparison with the applied forces, the resulting check (slit) may be described using the Christianovich model [1-4], developed in the context of the mechanics problems of hot rocks (for an evaluation of the limits of applicability of this cohesionless approximation to applied problems, see [5]).

In formulating the problem, the scheme of [4] is followed. Suppose that two half spaces with identical elastic properties (which may vary over the depth) are pressed together by a uniformly distributed stress $\sigma_{zz} = -\sigma$ (Fig. 1). Identical but opposite loads $q(\mathbf{r})$ tend to break the contact between these half spaces (such loads acting on the contour of the developing slit also result, as is known, from the above-mentioned volume forces disrupting the material [6, 7]). The displacements $\pm\, w(\mathbf{r}, \sigma)$ of the contours of the plane slit developing in the body, of unknown shape G_σ in plan, and the normal pressure on the half spaces composing the body $p(\mathbf{r}, \sigma) = -\sigma_z|_S$ must satisfy on S conditions in the form of alternating equalities and inequalities

$$\begin{aligned} p(\mathbf{r}, \sigma) &= q(\mathbf{r}) - \sigma, \quad w(\mathbf{r}, \sigma) > 0, \ \mathbf{r} \in G_\sigma, \\ p(\mathbf{r}, \sigma) &\geqslant q(\mathbf{r}) - \sigma, \quad w(\mathbf{r}, \sigma) = 0, \ \mathbf{r} \in S \setminus G_\sigma. \end{aligned} \tag{1}$$

Here and below, in view of the symmetry, the conditions are written only for the upper half space; $\mathbf{r} = (x, y)$ is a point of S. There are no tangential stresses nor cohesive forces at S. The inequalities in the conditions of Eq. (1) (unilateral constraints) reflect the physically clear requirement of "nonoverlapping" of the slit edges and the absence of a resulting tensile stress on the continuation of the slit — in the region of overlap of the half spaces (see also [8]). In the given formulation, this problem of the breakdown of an

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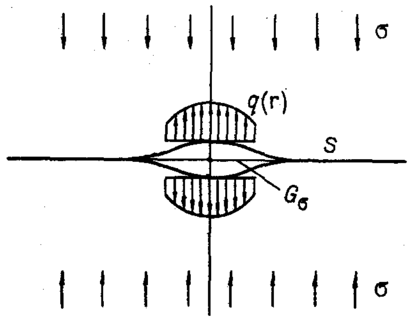

Fig. 1. Christianovich model of cohesionless crack formation.

elastic body not subjected to tensile stress over the S plane is a particular case of the Sin'orin problem [9] (on the loosening of an elastic half space away from a rigid base in ideal contact). The condition in Eq. (1) is "undetermined" (ambiguous) in Sin'orin—Fikera terminology. The existence and uniqueness of solutions of the Sin'orin problem has been demonstrated in [9]; certain estimates for a homogeneous half space are given in [8]. Note that for the given shape of the crack in plan (the slit) G, with the contour Γ, the displacements of the crack edges $w(\mathbf{r}, \Gamma)$ and the applied load $Q(\mathbf{r})$ are related by the pseudodifferential equation

$$\Lambda_G w(\mathbf{r}, \Gamma) = Q(\mathbf{r}). \tag{2}$$

In particular, for a crack in a homogeneous space of Young's modulus E and Poisson's ratio ν, the operator Λ_G takes the form [6]

$$\Lambda_G w(x, y) = -\frac{E}{4\pi(1-\nu^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\int\limits_G \frac{w(x', y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}. \tag{3}$$

An ideal representation of the expansion process of the Christianovich crack G_σ when the compressive pressure σ decreases from σ = ∞ (in which case the crack is closed) to $\sigma = \sigma_0$ will now be considered. Let Γ_σ be the contours of the corresponding regions of opening of the crack, for which Eq. (1) is satisfied. Then the functions

$$w_1(\mathbf{r}; \Gamma_\sigma) = -\frac{\partial w(\mathbf{r}, \sigma)}{\partial \sigma}, \quad p_1(\mathbf{r}; \Gamma_\sigma) = -\frac{\partial p(\mathbf{r}; \sigma)}{\partial \sigma} \tag{4}$$

are the values on S of the solution of the mixed problem with the conditions

$$p_1(\mathbf{r}; \Gamma_\sigma) = 1, \ \mathbf{r} \in G_\sigma; \ w_1(\mathbf{r}; \Gamma_\sigma) = 0, \ \mathbf{r} \in S \setminus G_\sigma. \tag{5}$$

Thus, $w_1(\mathbf{r}; \Gamma_\sigma)$ is the displacement of points of the contour of crack G_σ in the given body under the action of unit load (the solution of the equation $\Lambda_{G_\sigma} w_1(\mathbf{r}; \Gamma_\sigma) = 1$). Such solutions, called "unit solutions," are widely used in linear rupture mechanics, where they are usually treated from the viewpoint of the problem of a given crack in a homogeneous field of tensile forces [6, 7]. The following representation of the displacement $w(\mathbf{r}, \sigma)$ in the problem with the conditions in Eq. (1) in terms of the unit solutions for the slits G_τ with $\tau \geqslant \sigma$ is obtained from Eq. (5)

$$w(\mathbf{r}, \sigma) = \int\limits_\sigma^\infty w_1(\mathbf{r}; \Gamma_\tau)\,d\tau. \tag{6}$$

Equation (6) implies the property of smooth closure of the edges of the Christianovich crack on its contour when $q(\mathbf{r}) \geqslant -Q_0$: $w(\mathbf{r}, \sigma) = O(\rho_n^{3/2})$ in the vicinity of the contour Γ_σ of the growing crack, where ρ_n is the distance from the point $\mathbf{r} \in G_\sigma$ to Γ_σ.

Extremal Properties of Cohesionless-Slit Contours

The solution of the given problem has an important extremal property.* Of all the possible cracks G for a given load field $Q(\mathbf{r}) = q(\mathbf{r}) - \sigma$, the true contour Γ_σ of the Christianovich crack has the maximum volume of the resulting cavity

*This assertion was expressed in the form of the Barenblatt hypothesis in discussing [10].

$$V[\Gamma] = 2\int_G w(\mathbf{r}, \Gamma)\, ds, \quad \Gamma = \partial G, \tag{7}$$

i.e., finding the region of slit opening is equivalent to the problem of finding the maximum slit volume as a functional of the contour.

Before passing to a proof of this assertion, a convenient expression will be obtained for the volume $V[\Gamma;\ Q(\mathbf{r})]$ of the cavity forming in the space with the fixed plane crack G under the action of a normal load $Q(\mathbf{r})$ applied to the crack edges. Applying the reciprocity theorem to the stress–strain state corresponding to a crack of form G in plane for the given $Q(\mathbf{r})$ and unit load, it is found that

$$V[\Gamma;\ Q(\mathbf{r})] = 2\int_G Q(\mathbf{r})\, w_1(\mathbf{r};\ \Gamma)\, dr, \tag{8}$$

where $w_1(\mathbf{r};\ \Gamma)$ is the above-mentioned "unit" solution for the crack G.

In the case where a nonzero displacement $w = \pm\, U(\mathbf{r})$ is specified on the plane S except for the crack G (e.g., in the compression of half spaces along the common boundary of which there are rigid symmetrical inclusions), the expression for the volume associated with this crack may be written in the form

$$V[\Gamma] = 2\int_G Q(\mathbf{r})\, w_1(\mathbf{r};\ \Gamma)\, ds - 2\int_{S\setminus G} p_1(\mathbf{r};\ \Gamma)\, U(\mathbf{r})\, ds, \tag{9}$$

where $p_1(\mathbf{r};\ \Gamma) < 0,\ \mathbf{r} \in S\setminus G$ is the negative pressure (tensile stress) over the S plane outside the crack in the body with a unit crack of form G in plan. Equation (9) is also obtained by applying the reciprocity theorem to the unit solution $w_1(\mathbf{r};\ \Gamma)$, $p_1(\mathbf{r};\ \Gamma)$ and to the solution of the following mixed problem of a half space

$$\sigma_{zz}|_{z=0} = -Q(\mathbf{r}),\ \mathbf{r} \in G,\ w = U(\mathbf{r}),\ \mathbf{r} \in S\setminus G. \tag{10}$$

Equations (8) and (9) are analogous in form to the expressions for the force acting on a nonplane stamp of the given form pressed into the half space; these were obtained in [11].

Proof of Extremal Properties. Let Γ^0, be the crack contour, for which Eq. (1) is satisfied, and Γ' any other contour. Consider an auxiliary crack of form $G^1 = G^0 \cup G'$ in plan, the loads applied to the edges of this crack being equal to the pressures on the half space in this region corresponding to the solution of the problem on the crack G^0: $Q^1(\mathbf{r},\ G^1) = p^0(\mathbf{r}, \sigma)$.

Then the displacements of the edges of this crack coincide (the edges of the crack G^1 under the action of the load $p^0(\mathbf{r},\ \sigma)$ are superimposed in the subregion $G^1/G^0 = G'/G^0$), so that the volumes are also equal: $V^1 = V^0$. The expression for the volume V^1 is written in the form

$$V^1 = 2\int_{G'} w^0(\mathbf{r},\ \sigma)\, ds + 2\int_{G^0\setminus G'} w^0(\mathbf{r},\ \sigma)\, ds. \tag{11}$$

Equation (9) is now applied to the first term in Eq. (11), regarding it as the volume of a crack of form G' in plan, outside which the displacement $w^0(\mathbf{r},\ \sigma)$ of the half space surface is specified, while the load applied in G' is $Q'(\mathbf{r}) = p^0(\mathbf{r},\ \sigma)$:

$$2\int_{G'} w^0(\mathbf{r},\ \sigma)\, ds = 2\int_{G'} p^0(\mathbf{r},\ \sigma)\, w_1(\mathbf{r};\ \Gamma')\, ds - 2\int_{G^0\setminus G'} w^0(\mathbf{r},\ \sigma)\, p_1(\mathbf{r};\ \Gamma')\, ds. \tag{12}$$

The volume V' of the crack of form G' in plan under the action of a load $Q(\mathbf{r}) = q(\mathbf{r}) - \sigma$ is calculated from Eq. (8):

$$V' = 2\int_{G'} [q(\mathbf{r}) - \sigma]\, w_1(\mathbf{r};\ \Gamma')\, ds. \tag{13}$$

Combining Eqs. (11)-(13), the final result obtained for the difference in volume of cracks of different form — G^0 and G' — in plan under the action of the same field of specified forces $Q(\mathbf{r}) = q(\mathbf{r}) - \sigma$ is

$$V^0 - V' = 2\int\limits_{G^0\setminus G'} w^0(\mathbf{r},\ \sigma)\,[1 - p_1(\mathbf{r};\ \Gamma')]\,ds + 2\int\limits_{G'\setminus G^0} \{p^0(\mathbf{r},\ \sigma) - [q(\mathbf{r}) - \sigma]\}\,w_1(\mathbf{r};\ \Gamma')\,ds. \tag{14}$$

Taking into account that $w_1(r;\ \Gamma') \geqslant 0$ and $p_1(r;\ \Gamma') \leqslant 1$ if the operator Λ_G is positive (which is the case for a wide class of bodies [12]), and assuming that Eq. (1) is satisfied, Eq. (14) yields the necessary condition of the theorem: $V^0 \geqslant V'$. Since no constraint of inequality type was used in deriving Eq. (14), it also follows from Eq. (14) that the maximality condition in Eq. (7) is sufficient. In fact, suppose that a crack G^0 held open by the load $Q(r)$ is such that its volume V^0 is no less than the volume of any other crack V' in the same field of specified forces. Then G^0 is a Christianovich slit (since otherwise, when any of the conditions in Eq. (1) is satisfied, the region G' may be chosen such that $V' > V^0$).

Note that in the given proof the displacements of the edges of the "trial" crack G were not assumed to be nonnegative, and thus the "volume" V' expressed by Eqs. (7) and (13) must be understood in a formal algebraic sense.

Using the principle here proven significantly simplifies the use of Eq. (8), which for the problem in Eq. (1) may expediently be rewritten in the form

$$V[\Gamma,\ q(\mathbf{r}) - \sigma] = T[\Gamma;\ q(\mathbf{r})] - \sigma V_1[\Gamma], \tag{15}$$

$$V_1 = 2\int\limits_G w_1(\mathbf{r};\ \Gamma)\,ds, \qquad T = 2\int\limits_G q(\mathbf{r})\,w_1(\mathbf{r};\ \Gamma)\,ds. \tag{16}$$

If the extremal contour is sought in a space of crack boundaries which are specified by a finite number of parameters $\mu_1, \mu_2, \ldots, \mu_n$, the variational problem for the functional in Eq. (7) reduces to the system of equations

$$\frac{\partial T}{\partial \mu_k} - \sigma\,\frac{\partial V_1}{\partial \mu_k} = 0, \quad k = 1,\ 2,\ \ldots,\ n, \tag{17}$$

from which the geometrical parameters $\mu_1(\sigma), \ldots, \mu_n(\sigma)$, of the contour Γ_σ may also be found.

The formulation considered above naturally generalizes to the problem of the formation of a Christianovich crack in the symmetry plane S of a body of finite size under the action of identical but opposite forces normal to this plane (e.g., in the cross section of a cylinder).

The close relation between the approach here developed and the extremal methods of finding the unknown contact area when a stamp is pressed into an elastic body should be stressed. These problems with unilateral constraints bear a close resemblance.

In the case when there is an initial crack G_* in the symmetry plane of an elastic body, a related problem may be considered, the problem of finding the region of superposition of the crack edges (partial closure of the crack), under the assumption that the material does not break down, apart from the initial rupture [8, 13]. This problem coincides with the problem of a Christianovich crack whose region of opening may only extend within G_*, i.e., with constraints on the form of permissible contours. It is dual to the problem of pressing in a nonplane stamp bounded by a sharp edge, as considered in [10]. The desired (semiunknown) contour Γ^0 consists of sections Γ_*, continuously connected with the extremals of the functional $V[\Gamma]$.

Note that in the case when the contour Γ_σ belongs to a family of curves specified by a single geometrical parameter, the stationarity of the functional (function) in Eq. (7) with respect to this parameter may be established using just the representation of the solution as in Eq. (6) and Eq. (8). In fact, integrating both sides of Eq. (6) over the region G_σ, and taking into account that $G_\tau \subset G_\sigma$ when $\tau < \sigma$, it is found that

$$V[\sigma;\ \Gamma_\sigma] = \int\limits_\sigma^\infty V_1[\Gamma_\tau]\,d\tau. \tag{18}$$

If the contour Γ_σ is specified by a single parameter $a = a(\sigma)$ (e.g., the radius of a circle for disk-shaped cracks), so that $\sigma = \sigma(a)$ is a function incorporating the inverse dependence, Eqs. (15) and (18) yield two expressions for the volume V as a function of the variable a:

$$V(a) = -\int_0^a V_1(\alpha)\,\sigma'(\alpha)\,d\alpha = T(a;\ q) - \sigma(a)\,V_1(a). \tag{19}$$

Differentiating Eq. (19) with respect to a leads to an equation from which $a(\sigma)$ may be determined, identical with the stationarity condition in Eq. (17)

$$T'(a) - \sigma V_1'(a) = 0. \tag{20}$$

The given derivation of Eq. (20) closely resembles the corresponding discussion in finding the contact area in [14], while Eq. (6) serves as an analog of the expression for the contact stress on pressing in a stamp in terms of the solutions for "unit" stamps obtained in [14].

Having established these extremal properties, it is possible to isolate the intrinsically nonlinear part of the problem — finding an initially unknown slit contour Γ_σ corresponding to satisfaction of the unilateral conditions of no tensile stress and no overlapping of the edges — from the linear problem of determining the stress—strain state of a body with a crack of already known form in plan, which may be solved independently. In particular, the displacement of the edges is obtained using quadratures in Eq. (6).

No discussion will be given here of the aspects of the problem associated with the abstract theory of variational inequalities [9] and the extremal properties of elastic energy. The expression for the deformation energy of a body with a Christianovich crack G_σ (calculated from the energy of a compressed body without a crack) will simply be noted; it is obtained from Eq. (6)

$$W[\sigma;\ \Gamma_\sigma] = \int_\sigma^\infty V[\tau;\ \Gamma_\tau]\,d\tau. \tag{21}$$

Application of Variational Approach

Unfortunately, accurate solutions of "unit" problems are known only for circular and elliptical cracks [6]. Approximate solutions have also been obtained for cracks close to circular [6] and annular [15]. Methods of numerical solution of problems for cracks of arbitrary form in plan have been developed, based on the discretization of the corresponding integral or pseudodifferential equations by means of division of the crack region into cells and difference approximation of the solution [16, 17]. As will be shown, the problem in Eqs. (1) and (2) reduces, under these circumstances, to a finite-dimensional extremal problem.

Example 1. An Elliptical Crack in a Homogeneous Elastic Space Held Open by a Load $q(x,\ y) = Q_0\,\delta(y)$ Applied over the Segment $|x| \leqslant l,\ y = 0$. At small values of the dimensionless load parameter $N = Q_0/\sigma l$, the contour of a Christianovich crack is a narrow ellipse close to the line of action of the load; for large N, it approaches a circle corresponding to a pair of point forces $2Q_0 l$ applied at the points $x = y = 0$. Therefore, the contours Γ_σ will be looked for in the family of ellipse with constant focal distance $2l$ and a major semiaxis a.* The unit solution for such cracks is of the form

$$w_1(x,\ y) = \frac{\sqrt{1 - l^2/a^2}}{DE(l/a)}\sqrt{a^2 - x^2 - \frac{y^2}{1 - l^2/a^2}},$$

where $E(l/a)$ is the total elliptical integral; $D = E/2(1 - \nu^2)$ is the elastic constant. Calculating Eq. (16), the extremum condition in Eq. (17) yields an equation relating the contour eccentricity $e = l/a$ and the load parameter N:

$$\frac{3N}{2\pi}\left\{\frac{1+e^2}{e^2} + \left(\frac{3\sqrt{1-e^2}}{e^3} + \frac{1}{e\sqrt{1-e^2}}\right)\arcsin e\right\} - \frac{\mathrm{K}(e)}{\mathrm{E}(e)}\left(\frac{1-e^2}{e^2} + \frac{\sqrt{1-e^2}}{e^3}\arcsin e\right) = \frac{4}{e^4} - \frac{2}{e^2} - \frac{\mathrm{K}(e)}{\mathrm{E}(e)}\left(\frac{1}{e^4} - \frac{1}{e^2}\right). \tag{22}$$

*More detailed analysis shows that this assumption is satisfied with high accuracy.

TABLE 1. Dimensionless of Annular Crack as a Function of Dimensionless Load $M = P_0/\pi\sigma c^2$

M	0	0,5	1,0	2,0	2,5	2,8	3,0	3,1	3,149
a/c	1,0	0,918	0,832	0,603	0,449	0,3285	0,216	0.125	0
b/c	1,0	1,082	1,168	1,310	1,383	1,424	1,450	1.462	1,467

Example 2. An Axisymmetric Crack Held Open by a Load Applied over the Circumference of a Circle of Radius c, Where $c(q(\rho)=P_0\delta(\rho-c)/2\pi\rho)$. The slit which appears is disk-shaped only at sufficiently large values of the dimensionless load parameter $M=P_0/\pi\sigma c^2>M_*$. At small M, the crack is annular. Using the asymptotic solutions obtained in [15] for a crack that is annular in plan $a\leqslant\rho\leqslant b$, under the action of unit load, the initially unknown slit dimensions may be determined as a function of M. For example, for small M, the ring is narrow ($b/a-1\ll 1$), and the corresponding unit solution may be written in the following asymptotic form

$$w_1(\rho)=\frac{a^{3/4}b^{5/4}}{D\rho^2}\sqrt{(\rho-a)(b-\rho)}\,[1+(0.246+\ln\lambda/16)\lambda^{-2}+1.75(\rho/\sqrt{ab}-1)], \tag{23}$$

$$\lambda=2a/(b-a).$$

The values of a and b are determined from Eq. (17), where $\mu_1 = a$, $\mu_2 = b$. In the approximation adopted, $a/c = 1-\varepsilon$, $b/c = 1+\varepsilon$, where $\varepsilon\ll 1$, and the following equation, accurate up to terms of order ε^3, is satisfied:

$$\pi\varepsilon[2+1.5\varepsilon+1.675\varepsilon^2-(0.25\varepsilon^2+0.3125\varepsilon^3)\ln\varepsilon]$$
$$=M[1+\varepsilon+0.6755\varepsilon^2+2.0328\varepsilon^3-(0.1875\varepsilon^2+0.125\varepsilon^3)\ln\varepsilon]. \tag{24}$$

The values of a/c and b/c for $M\leqslant 1$ are given in Table 1. For $M < M_*$, but close to this critical load parameter, the ring is broad ($b/a\gg 1$). The unit solution in this case has the simple asymptotic form [15]

$$w_1(\rho)=\frac{4}{\pi^2 D}\sqrt{b^2-\rho^2}\arccos\frac{a}{\rho}.$$

The extremum condition in Eq. (17) gives a system of equations for determining the crack dimensions

$$\frac{2M}{\pi}\sqrt{\frac{b^2-c^2}{c^2-a^2}}-\frac{b^2-a^2}{c^2}=0,$$
$$\frac{M}{2}\frac{b}{\sqrt{b^2-c^2}}\arccos\frac{a}{c}-\int_a^b\frac{b\rho}{\sqrt{b^2-\rho^2}}\arccos\frac{a}{\rho}\,d\rho=0. \tag{25}$$

Numerical solution of Eq. (25) gave the values of a/c and b/c for $M\geqslant 2$ given in Table 1. The critical value of M is $M_*\approx 3.149$.

When $M > M_*$, the crack is disk-shaped, and its radius b is determined from the second relation in Eq. (25) with $a = 0$, which coincides with the result obtained from the condition that the stress-intensity factor vanishes on the contour

$$2b_1\sqrt{b_1^2-1}=M,\ b_1=b/c. \tag{26}$$

In the course of proving the extremal properties, it has already been noted that the true contour corresponds to the absolute maximum of the crack volume. For this to be the case, of course, the chosen system of parameters specifying the trial contours must be sufficiently complete. If, for example, the form of the load is not taken into account in the given problem, that of finding the solution for any M in the class of disk-shaped cracks, overlapping of the crack edges is obtained near the center when $M < M_*$ in circular cracks corresponding to the formal solution of Eq. (26), and the volume is smaller than for a true annular slit.

Example 3. Numerical Solution of the Problem of a Slit with an Unknown Contour. Let the region S_1, which is known to contain the region of opening of the crack G^0, be divided into cells (i). Using any numerical method of solving Eq. (2) in S_1, e.g., the variational–difference approach [17], Eq. (2) is reduced to a system of linear equations for the unknown displacements in the cells

$$\sum_{j=1}^{N} A_{ij} w_j = p_i, \quad i = 1, \ldots, N. \tag{27}$$

The desired region G^0 corresponds to the part of S_1 in which the load on the half space coincides with the external forces holding the crack open

$$\sum_{j=1}^{N} A_{ij} w_j = Q_i, \quad (i) \in G^0,$$

while in the subregion $S_1 \setminus G^0$ the edges of the crack S_1 are superimposed:

$$w_i = 0, \quad (i) \in S_1 \setminus G^0.$$

The next step is to pass, in accordance with the statement of the theorem, to the following finite-dimensional problem on the extremum condition: to maximize

$$V = \sum_{i=1}^{N} s_i w_i \tag{28}$$

(where s_i are the areas corresponding to the expression for the elementary volume in terms of the variables in cell i) under the constraints

$$\Big[\sum_{j=1}^{N} A_{ij} w_j - Q_i\Big] w_i = 0, \quad i = 1, \ldots, N. \tag{29}$$

Any of the methods of directed search may be used to solve the resulting extremal problem graphically (the set consisting of a finite but very large number of points in N-dimensional space — the solutions of Eq. (29) corresponding to different subregions $G \subset S_1$). However, the following iterative method for the approximate solution of Eqs. (28) and (29) is more effective. Suppose that $p^k = (p_1^k, \ldots, p_N^k)$ is the load vector on the half spaces comprising the body at the selected grid points and $w^k = (w_1^k, \ldots, w_N^k)$ the displacement of the crack edges at these points, in the k-th iteration. At the points i where $p_i^k < Q_i$, set $w_i^{k+1} = w_i^k + \Delta$. The varying pressure on the half spaces is calculated in accordance with Eq. (27): $p^{k+1} = Aw^{k+1}$, and then the procedure is repeated. Complete closures of the crack may be taken as the initial approximation: $w_i^0 = 0$, $i = 1, \ldots, N$.

Since the displacements and hence the integral function in Eq. (28) do not decrease in this process, the iteration converges, after a finite number of steps, to the approximate solution w_* of the problem of crack-edge superposition. In a symmetric matrix of coefficients, the diagonal elements are positive, while the off-diagonal elements are negative and much smaller in absolute magnitude, so that $A_{ii} > \sum_{j \neq i} |A_{ji}|$. Therefore, the iterational process is stable, and the approximate solution w_* satisfies the condition

$$0 < p_i^* - Q_i < \varepsilon \text{ when } w_i^* > 0,$$

where the discrepancy in the stress $\varepsilon < (\max A_{ii})\Delta$, and may be made sufficiently small by reducing the displacement step Δ.

Note, in conclusion, that the approach here proposed may be directly examined to thermoelastic problems and problems on cracks in elastic–porous media saturated with liquid, in the presence of filtrational forces, when the additional loads due to heat and mass transfer are independent of the strain.

NOTATION

S, rupture plane; σ, compression-force intensity; Γ, cut (crack) contour; p(r), normal load on S; Λ_G, pseudodifferential operator on G; $V[\Gamma]$, cavity volume corresponding to the crack G; $\mu_1, \ldots, \mu_n$, geometrical parameters of contour Γ; E, Young's modulus; ν, Poisson's ratio; $\delta(y)$, Dirac delta function; A_{ij}, matrix coefficients for discretization (1, 2); $p = (p_1, \ldots, p_N)$, vector of loads at grid points; $w = (w_1, \ldots, w_N)$, vector of displacements at grid points; ρ, polar radius.

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